Differential Geometry and its Applications 4 (1994) 239–258
North-Holland

Yamabe metrics of positive scalar curvature and conformally flat manifolds

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Communicated by J.P. Bourguignon
Received 6 February 1993

Abstract: Let $\mathcal{CY}(n,\mu_0,R_0)$ be the class of compact connected smooth manifolds M of dimension $n \geqslant 3$ and with Yamabe metrics g of unit volume such that each (M,g) is conformally flat and satisfies

$$\mu(M,[g]) \geqslant \mu_0 > 0, \qquad \int_M |E_g|^{n/2}\,dv_g \leqslant R_0,$$

where $[g]$, $\mu(M,[g])$ and E_g denote the conformal class of g, the Yamabe invariant of $(M,[g])$ and the traceless part of the Ricci tensor of g, respectively. In this paper, we study the boundary $\partial\mathcal{CY}(n,\mu_0,R_0)$ of $\mathcal{CY}(n,\mu_0,R_0)$ in the space of all compact metric spaces equipped with the Hausdorff distance. We shall show that an element in $\partial\mathcal{CY}(n,\mu_0,R_0)$ is a compact metric space (X,d). In particular, if (X,d) is not a point, then it has a structure of smooth manifold outside a finite subset $\mathcal{S}$, and moreover, on $X\backslash\mathcal{S}$ there is a conformally flat metric g of positive constant scalar curvature which is compatible with the distance d.

Keywords: Yamabe metric, positive scalar curvature, flat conformal structure.

MS classification: 53C; 58E.

1. Introduction

Let M be a compact connected smooth manifold of dimension n and $\mathcal{M}(M)$ denote the space of all smooth Riemannian metrics on M. Throughout this paper, we always assume that manifolds under consideration are connected, smooth and that the dimension n is at least 3. It is a classical result due to Hilbert [14] that a metric h on M is an *Einstein metric*, i.e. a metric whose Ricci curvature tensor is proportional to the metric tensor, if and only if h is a critical point for the functional of the normalized total scalar curvature $\mathcal{R}$ defined as

$$\mathcal{R}(g) = \frac{\int_M R_g\,dv_g}{\left(\int_M dv_g\right)^{(n-2)/n}} \qquad \text{for} \quad g \in \mathcal{M}(M),$$

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[1]This research was partially supported by the Grant-in-Aid for Young Scientists No. 06854003, The Ministry of Education, Science and Culture.

SSDI 0926-2245 (94) 00015-8

where R_g and dv_g denote the scalar curvature and the volume element of g, respectively. An Einstein metric may be considered as a good candidate for a metric which makes a given manifold M the best possible shape. It is, however, known that there exist compact manifolds which carry no Einstein metric (e.g. $S^1 \times S^2$ and $S^1 \times S^3$, see [6, Chapter 6]). Nevertheless, $S^1 \times S^2$ and $S^1 \times S^3$ may admit positive constant curvature metrics (and so Einstein metrics) with suitable degenerations (see Example 1.1 below). Therefore it is reasonable to expect the existence of a suitable degenerate Einstein metric on a given manifold M.

On the other hand, the *Yamabe functional* I on a conformal class C (not necessarily conformally flat) of M is defined by the restriction of $\mathcal{R}$ to C

$$I(g) = (\mathcal{R}|_C)(g) \qquad \text{for} \quad g \in C.$$

The infimum of this functional is denoted by $\mu(M,C)$, i.e.,

$$\mu(M,C) = \inf_{g \in C} I(g)$$

and called the *Yamabe invariant* of (M,C). The following so-called Yamabe problem was solved affirmatively by the work of Yamabe [34], Trudinger [33], Aubin [3] and Schoen [29,32]:

Given a conformal class C on a compact manifold M of dimension $n \geqslant 3$, find a metric g which minimizes the Yamabe functional I on the conformal class C.

We call a metric, which is a solution of the Yamabe problem, simply a *Yamabe metric.*

A conformal class C on M is called a *flat conformal structure* if (M,g) is conformally flat for $g \in C$. Also, a Riemannian metric g on M will be called a *conformally flat Yamabe metric* if (M,g) is conformally flat and g is a Yamabe metric. From now on we assume that M is a compact manifold which admits a flat conformal structure.

In this paper, we are concerned with the problem of finding a suitable degenerate conformally flat Einstein metric (i.e. constant curvature metric) on M. The Yamabe problem can be viewed as a first approach for this problem. As a second approach, we shall consider an invariant $\mu_C(M)$ of M defined to be the supremum of $\mu(M,C)$ of all flat conformal structures C on M, i.e.,

$$\mu_C(M) = \sup\{\mu(M,C); C \text{ is a flat conformal structure on } M\}.$$

It should be pointed out that the invariant $\mu_C(M)$ was introduced by Izeki [16] as an analogy of the invariant

$$\mu(N) = \sup\{\mu(N,C); C \text{ is a conformal class on } N\},$$

due to Kobayashi [20] and Schoen [31], defined for compact manifolds N possibly with no flat conformal structure. The invariant $\mu(N)$ was introduced under the computation of the second variation of the functional $\mathcal{R}$ at an Einstein metric and the observation of the distinction between conformal directions and their orthogonal complements (see Koiso [22] and [21,31] for details).

Now we can formulate our problem in a naive form (cf. [2]):

Problem P. *Let $\{g_i\}_{i=1}^{\infty}$ be a sequence of conformally flat Yamabe metrics on M with unit volume which satisfies*

$$\lim_{i\to\infty} \mu(M,[g_i]) = \mu_C(M).$$

Then, characterize the degeneration of the limit of $\{(M,g_i)\}_{i=1}^{\infty}$ as $i \to \infty$.

Before we state our main theorem, let us look at a typical example due to Kobayashi [19] and Schoen [31].

Example 1.1. There exists a family of conformally flat Yamabe metrics $\{g_t\}_{t\geqslant t_0}$ on $S^1 \times S^{n-1}$ such that

(1) $[g_{t_1}]$ and $[g_{t_2}]$ are not equivalent if $t_2 > t_1 \geqslant t_0$.

(2) $(S^1 \times S^{n-1}, g_t)$ converges to the quotient space of $S^n(1)$ which identifies its north pole p^+ with its south pole p^-, in the Hausdorff distance, where $S^n(1)$ denotes the Euclidean n-sphere of radius 1.

(3) There exist $q_0 \in S^1$ and a Riemannian metric g_∞ on $S^1 \times S^{n-1}$ which degenerates on $\{q_0\} \times S^{n-1}$ such that $(S^1 \times S^{n-1} \backslash (\{q_0\} \times S^{n-1}), g_\infty)$ is isometric to $S^n(1) \backslash \{p^+, p^-\}$.

(4) $\lim_{t\to\infty} \mu(S^1 \times S^{n-1}, [g_t]) = \mu_C(S^1 \times S^{n-1}) = \mu(S^n) = n(n-1)\mathrm{vol}(S^n(1))^{2/n}$.

(5) $\lim_{t\to\infty} \mathrm{vol}(S^1 \times S^{n-1}, g_t) = \mathrm{vol}(S^n(1))$.

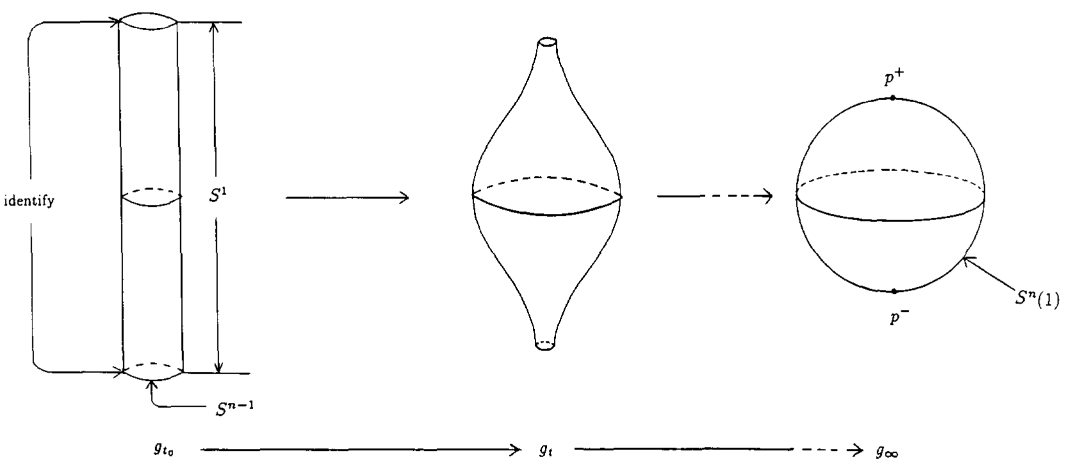
(6) $\lim_{t\to\infty}\{\inf\{\mathrm{Ric}_{g_t}(x); x \in S^1 \times S^{n-1}\}\} = -\infty$, where Ric_{g_t} denotes the Ricci curvature of g_t.

(7) $\lim_{t\to\infty} i(S^1 \times S^{n-1}, g_t) = 0$, where $i(S^1 \times S^{n-1}, g_t)$ denotes the injectivity radius of $(S^1 \times S^{n-1}, g_t)$.

(8) $\lim_{t\to\infty} \int_{S^1\times S^{n-1}} |E_{g_t}|^{n/2} dv_{g_t} < +\infty$.

(9) For any $p > n/2$, $\lim_{t\to\infty} \int_{S^1\times S^{n-1}} |E_{g_t}|^{p} dv_{g_t} = +\infty$.

(10) If $n = 3$, for $q_1 \in S^2$ and each $t \geqslant t_0$ there exists $r_t > 0$ such that $\lim_{t\to\infty} r_t = \infty$ and $((S^1 \times S^2, r_t g_t), (q_0, q_1))$ converges to the double of M_S in the pointed Hausdorff distance, where M_S denotes the space section of the outside region of Schwarzschild's solution in general relativity (cf. [13]).



Inspired by the above example, we prove the following convergence theorem as a one step for the Problem P.

Theorem 1.2. *Let $\{(M_i, g_i)\}_{i=1}^{\infty}$ be a sequence in $\mathcal{CY}(n, \mu_0, R_0)$ with some positive constants μ_0, R_0. Then either of the following two cases must be hold.*

(1°) *(M_i, g_i) converges to a point in the Hausdorff distance.*

(2°) *There exist a subsequence $\{j\} \subset \{i\}$, a compact connected metric space (M_∞, d_∞) with positive diameter, and a finite subset $\mathcal{S} = \{x_1, \ldots, x_k\} \subset M_\infty$ (possibly empty) such that*

(2°.1) *(M_j, g_j) converges to (M_∞, d_∞) in the Hausdorff distance.*

(2°.2) *$M_\infty \setminus \mathcal{S}$ has a structure of smooth n-manifold and a conformally flat metric g_∞ of positive constant scalar curvature which is compatible with the distance d_∞ on $M_\infty \setminus \mathcal{S}$.*

(2°.3) *For each compact subset $K \subset M_\infty \setminus \mathcal{S}$, there exists an into diffeomorphism $\Phi_j : K \longrightarrow M_j$ for j sufficiently large such that $(\Phi_j)^* g_j$ converges to g_∞ in the C^∞ topology on K.*

(2°.4) $\lim_{j \to \infty} R_{g_j} = R_{g_\infty}$.

(2°.5) *For every $x_a \in \mathcal{S}(a = 1, \ldots, k)$ and j, there exist $x_{a,j} \in M_j$ and positive number r_j such that*

(2°.5a) *$B_r(x_{a,j})$ converges to $B_r(x_a)$ in the Hausdorff distance for all $r > 0$.*

(2°.5b) $\lim_{j \to \infty} r_j = \infty$.

(2°.5c) *$((M_j, r_j g_j), x_{a,j})$ converges to $((N_a, h_a), x_{a,\infty})$ in the pointed Hausdorff distance, where (N_a, h_a) is a complete noncompact, scalar flat, conformally flat, non-flat n-manifold which satisfies*

$$\sup_{N_a} |\mathrm{Ric}_{h_a}| < \infty, \quad 0 < \int_{N_a} |\mathrm{Ric}_{h_a}|^{n/2} \, dv_{h_a} < \infty,$$

and

$$\mathrm{vol}(B_r(p), h_a) \geqslant (5 \cdot 2^n (n-1))^{-n/2} (n-2)^{n/2} \mu_0^{n/2} r^n$$

for $p \in N_a$ and $r > 0$.

(2°.5d) *For every $r > 0$, there exists an into diffeomorphism $\Psi_j : B_r(x_{a,\infty}) \longrightarrow M_j$ for j sufficiently large such that $(\Psi_j)^*(r_j g_j)$ converges to h_a in the C^∞ topology on $B_r(x_{a,\infty})$.*

(2°.6) *It holds*

$$\lim_{j \to \infty} \int_{M_j} \left| E_{g_j} \right|^{n/2} dv_{g_j} \geqslant \int_{M_\infty} \left| E_{g_\infty} \right|^{n/2} dv_{g_\infty} + \sum_{a=1}^{k} \int_{N_a} \left| \mathrm{Ric}_{h_a} \right|^{n/2} dv_{h_a}.$$

The author believes that under the assumption of Theorem 1.2, only the second case (2°) holds.

Remark 1.3. (1) In [32], Schoen and Yau proved that if (M, C) is a conformally flat compact manifold of positive Yamabe invariant, then (M, C) is a Kleinian manifold. Hence, an element in $\mathcal{CY}(n, \mu_0, R_0)$ is also a Kleinian manifold.

(2) For $p > n/2$, let $\mathcal{CY}_p(n, \mu_0, R_0)$ denote the class of compact n-manifolds M with conformally flat Yamabe metrics g of unit volume which satisfy

$$\mu(M, [g]) \geqslant \mu_0 > 0, \int_M |E_g|^p \, dv_g \leqslant R_0.$$

In Theorem 1.2, if we replace the set $\mathcal{CY}(n, \mu_0, R_0)$ by the set $\mathcal{CY}_p(n, \mu_0, R_0)$ for $p > n/2$, then the conclusion which is the same as the first case (1°) never hold. In fact, from the result for volume estimates of geodesic balls from above due to Yang [35] (see also Gallot [9]), there exists a positive constant $D_0 = D_0(n, p, \mu_0, R_0)$ such that

$$\mathrm{diam}(M, g) \geqslant D_0 > 0 \quad \text{for} \quad (M, g) \in \mathcal{CY}_p(n, \mu_0, R_0).$$

However, from the viewpoint of Example 1.1, this set $\mathcal{CY}_p(n, \mu_0, R_0)$ may not be a suitable class for the Problem P.

(3) Our main Theorem 1.2 was also inspired by the convergence theorems for Einstein metrics, which are due to Anderson [1] and Nakajima [25] (also [5]).

The author would like to thank Professor O. Kobayashi for his continuous valuable advices and Professor H. Izeki for useful discussions. He would also like to thank Professor S. Nishikawa and the referee for many helpful suggestions.

2. Geometric inequalities for Yamabe metrics

In this section we shall give new geometric inequalities for Yamabe metrics of positive scalar curvature and prove, as an application, a precompactness theorem for them.

First, we discuss several known properties for Yamabe metrics. Let M be a compact n-manifold. Throughout this section, we do not assume that M admits a flat conformal structure. Since a Yamabe metric g (not necessarily of positive Yamabe invariant) on M is a minimizer of the Yamabe functional $I : [g] \longrightarrow \mathbb{R}$, then the first and second variational formulas show the following properties (cf. [4, 24]):

$$R_g = \mu(M, [g]) \mathrm{vol}(M, g)^{-2/n} = \text{const.}, \tag{2.1}$$

$$\lambda_1(-\Delta_g) \geqslant \frac{R_g}{n-1}, \tag{2.2}$$

where $\Delta_g = g^{ij} \nabla_i \nabla_j$ denotes the (nonpositive) Laplacian of (M, g), ∇ the Levi-Civita connection of (M, g) and $\lambda_1(-\Delta_g)$ the first nonzero eigenvalue of $-\Delta_g$. Moreover, the following property is due to Aubin [3].

$$\mu(M, [g]) \leqslant n(n-1) \mathrm{vol}(S^n(1))^{2/n}. \tag{2.3}$$

Now we state a crucial inequality for Yamabe metrics of positive scalar curvature.

Proposition 2.1. *Let g be a Yamabe metric of positive scalar curvature on M. Then*

$$\mathrm{vol}(B_r(p)) \geqslant (5 \cdot 2^{n-2})^{-n/2} c_g^{n/2} r^n \tag{2.4}$$

for $p \in M$ *and*

$$r \leqslant \sqrt{\frac{\operatorname{vol}(M,g)^{2/n}}{c_g}},$$

where $B_r(p)$ *denotes the geodesic ball of radius* r *centered at* p, $\operatorname{vol}(B_r(p)) = \operatorname{vol}(B_r(p), g)$ *the volume of* $B_r(p)$ *with respect to* g *and*

$$c_g = \frac{n-2}{4(n-1)}\mu(M,[g]) > 0.$$

Proof. The Yamabe invariant $\mu(M,[g])$ is rewritten as

$$\mu(M,[g]) = \inf_{\substack{u\in H_1(M)\\ u\not\equiv 0}} \frac{4\frac{n-1}{n-2}\int_M |du|^2\,dv_g + \int_M R_g u^2\,dv_g}{\left(\int_M |u|^{2n/(n-2)}\,dv_g\right)^{(n-2)/n}}, \tag{2.5}$$

where $H_1(M)$ denotes the Sobolev space of functions with L^2 first derivatives (cf. [4]). It then follows from (2.1), (2.5) and the positivity of $\mu(M,[g])$ that

$$\begin{aligned}&\left(\int_M |u|^{2n/(n-2)}\,dv_g\right)^{(n-2)/n}\\ &\qquad\leqslant \frac{1}{c_g}\int_M |du|^2\,dv_g + \frac{1}{\operatorname{vol}(M,g)^{2/n}}\int_M u^2\,dv_g\end{aligned} \tag{2.6}$$

for $u \in H_1(M)$.

Take any point $p \in M$, any $r > 0$ satisfying $r \leqslant \sqrt{\operatorname{vol}(M,g)^{2/n}/c_g}$ and fix them. Let

$$\zeta(x) = \begin{cases} 1 & (0 \leqslant \rho(x) \leqslant \frac{1}{2}r)\\ 2 - \dfrac{2}{r}\rho(x) & (\frac{1}{2}r \leqslant \rho(x) \leqslant r)\\ 0 & (r \leqslant \rho(x))\end{cases} \quad \text{for } x \in M,$$

where $\rho(x) = \operatorname{dist}_{(M,g)}(p,x)$. Then $\zeta \in H_1(M)$ and a straightforward calculation using the inequality (2.6) yields

$$\begin{aligned}\operatorname{vol}(B_{\frac{r}{2}}(p))^{(n-2)/n} &\leqslant \frac{4}{r^2 c_g}\{\operatorname{vol}(B_r(p)) - \operatorname{vol}(B_{\frac{r}{2}}(p))\}\\ &\quad + \frac{1}{\operatorname{vol}(M,g)^{2/n}}\operatorname{vol}(B_r(p)) \leqslant \frac{5}{r^2 c_g}\operatorname{vol}(B_r(p)).\end{aligned}$$

This gives the estimates

$$V_r \geqslant \tfrac{1}{5}c_g r^2 (V_{\frac{r}{2}})^{\beta}, \tag{2.7}$$

where $V_r = \operatorname{vol}(B_r(p))$ and $\beta = (n-2)/n$ (< 1).

This inequality (2.7) may now be iterated in the standard fashion. Let $r_1 = r$ and $r_m = 2^{-m+1}r$. We then obtain

$$\begin{aligned} V_r &\geqslant (5^{-1}c_g r^2)^{\sum_{i=1}^{m}\beta^{i-1}} 2^{-2\sum_{j=1}^{m-1} j\beta^j} (V_{2^{-m}r})^{\beta^m} \\ &\geqslant (5^{-1}c_g r^2)^{\frac{1-\beta^m}{1-\beta}} 2^{\frac{-2\beta}{(1-\beta)^2}} (V_{2^{-m}r})^{\beta^m}. \end{aligned}$$

Since $V_t = \omega_n t^n + O(t^{n+2})$ for small $t > 0$, letting $m \to \infty$ gives

$$V_r \geqslant (5 \cdot 2^{n-2})^{-n/2} c_g^{n/2} r^n,$$

where ω_n denotes the volume of the Euclidean n-ball of radius 1. This completes the proof of Proposition 2.1.

Proposition 2.2. *Under the same assumption of Proposition* 2.1,

$$\operatorname{diam}(M,g) \leqslant 2(5 \cdot 2^{n-2})^{n/2} \sqrt{\frac{\operatorname{vol}(M,g)^{2/n}}{c_g}}. \tag{2.8}$$

Proof. We may assume that $\operatorname{diam}(M,g) > \sqrt{\operatorname{vol}(M,g)^{2/n}/c_g}$, otherwise $\operatorname{diam}(M,g) \leqslant \sqrt{\operatorname{vol}(M,g)^{2/n}/c_g}$ and so our assertion is true. Let $d_0 = \sqrt{\operatorname{vol}(M,g)^{2/n}/c_g}$ and $n_0 = [\operatorname{diam}(M,g)/2d_0] + 1$, where $[a]$ denotes the greatest integer k satisfying $k \leqslant a$. Then there exists a disjoint family of geodesic balls $\{B_{d_0}(x_j)\}_{j=1}^{n_0}$ of radius d_0. It follows from (2.4) that

$$n_0(5 \cdot 2^{n-2})^{-n/2} c_g^{n/2} d_0^n \leqslant \sum_{j=1}^{n_0} \operatorname{vol}(B_{d_0}(x_j)) \leqslant \operatorname{vol}(M,g).$$

Since

$$\frac{\operatorname{diam}(M,g)}{2d_0} \leqslant \left[\frac{\operatorname{diam}(M,g)}{2d_0}\right] + 1 = n_0,$$

we then obtain

$$\begin{aligned} \operatorname{diam}(M,g) &\leqslant 2d_0 n_0 \\ &\leqslant 2(5 \cdot 2^{n-2})^{n/2} c_g^{-n/2} d_0^{-(n-1)} \operatorname{vol}(M,g) \\ &\leqslant 2(5 \cdot 2^{n-2})^{n/2} \sqrt{\frac{\operatorname{vol}(M,g)^{2/n}}{c_g}}. \end{aligned}$$

This completes the proof of Proposition 2.2.

Next, we recall the definition of the Hausdorff distance on the set $\mathcal{MET}$ of all isometry classes of compact metric spaces introduced by Gromov [12]. Let X and Y be compact metric spaces. A map $f : X \longrightarrow Y$ (not necessarily continuous) is said to be an ε-*Hausdorff approximation* if the following two conditions are satisfied.

$$\text{The } \varepsilon\text{-neighborhood of } f(X) \text{ in } Y \text{ is equal to } Y. \tag{2.9}$$

$$\left|d_X(p,q) - d_Y(f(p),f(q))\right| < \varepsilon \quad \text{for } p,q \in X. \tag{2.10}$$

The *Hausdorff distance* $d_H(X,Y)$ between X and Y is defined to be the infimum of all positive numbers ε such that there exist ε-Hausdorff approximations from X to Y and from Y to X. Unfortunately $d_H(\cdot,\cdot)$ does not satisfy the triangle inequality. However the inequality (2.11) below holds and then shows that it gives a metrizable complete uniform structure on the set $\mathcal{MET}$. Thus we treat $d_H(\cdot,\cdot)$ as if it is a distance function.

$$d_H(X,Z) \leqslant 2\{d_H(X,Y)+d_H(Y,Z)\}, \tag{2.11}$$

for $X,Y,Z \in \mathcal{MET}$.

For noncompact metric spaces, we also recall the definition of the pointed Hausdorff distance. Let (X,x) and (Y,y) be pointed metric spaces (possibly compact). A map $f:(X,x) \longrightarrow (Y,y)$ is said to be an *ε-pointed Hausdorff approximation* if

$$f(x)=y, \tag{2.12}$$

$$f(B_{1/\varepsilon}(x)) \subset B_{1/\varepsilon}(y), \tag{2.13}$$

$$f|_{B_{1/\varepsilon}(x)} : B_{1/\varepsilon}(x) \longrightarrow B_{1/\varepsilon}(y) \text{ is an } \varepsilon\text{-Hausdorff approximation.} \tag{2.14}$$

The *pointed Hausdorff distance* $d_{p,H}((X,x),(Y,y))$ between pointed metric spaces (X,x) and (Y,y) is the infimum of all positive number ε such that there exist ε-pointed Hausdorff approximations from (X,x) to (Y,y) and from (Y,y) to (X,x). $d_{p,H}(\cdot,\cdot)$ also defines a distance on the set $\mathcal{MET}_0$ of all isometry classes of pointed metric spaces whose metric balls are all precompact.

Let $\mathcal{Y}(n,\mu_0)$ denote the class of compact n-manifolds M with Yamabe metrics g of unit volume which satisfy

$$\mu(M,[g]) \geqslant \mu_0 > 0.$$

Theorem 2.3. *The set $\mathcal{Y}(n,\mu_0)$ is precompact in $\mathcal{MET}$ with respect to the Hausdorff distance.*

Proof. Using the same argument as in Gromov [12, Chapter 5] and the diameter estimate (2.8), it suffices to prove for each ε (>0) there exists a positive integer $N(\varepsilon)$, which is independent of $(M,g) \in \mathcal{Y}(n,\mu_0)$, such that

$$\frac{\mathrm{vol}(M,g)}{\mathrm{vol}(B_\varepsilon(p),g)} \leqslant N(\varepsilon) \quad \text{for any } p \in M. \tag{2.15}$$

The estimate (2.15) easily follows from (2.3) and (2.4). This completes the proof of Theorem 2.3.

3. A local a priori estimate of curvature

In this section we shall derive a local pointwise curvature estimate of conformally flat Yamabe metrics with positive scalar curvature.

Lemma 3.1. *Let g be a conformally flat Yamabe metric of nonnegative scalar curvature on a compact manifold M. Then,*

$$|\mathrm{Riem}_g| \leqslant \left\{\frac{4}{n-2}|E_g|^2 + 2n(n-1)\Big(\frac{\mathrm{vol}(S^n(1))}{\mathrm{vol}(M,g)}\Big)^{4/n}\right\}^{1/2}, \tag{3.1}$$

where Riem_g *denotes the Riemann curvature tensor of* g.

Proof. Since the Weyl tensor W_g of g vanishes identically, we have the following identity

$$|\mathrm{Riem}_g|^2 = \frac{4}{n-2}|E_g|^2 + \frac{2}{n(n-1)}R_g^2.$$

It then follows from (2.1), (2.3) and the nonnegativity of $\mu(M,[g])$ that

$$|\mathrm{Riem}_g|^2 \leqslant \frac{4}{n-2}|E_g|^2 + 2n(n-1)\Big(\frac{\mathrm{vol}(S^n(1))}{\mathrm{vol}(M,g)}\Big)^{4/n}.$$

The basic idea of the following estimate is originally due to Sacks-Uhlenbeck [28] (see also Anderson [1] and Schoen [30]).

Proposition 3.2. *Let g be a conformally flat Yamabe metric of positive scalar curvature on M. Then there exist positive constants $c_1 = c_1(n,\mu(M,[g]),\mathrm{vol}(M,g))$, $c_2 = c_2(n)$ and $\varepsilon_0 = \varepsilon_0(n,\mu(M,[g]))$ such that if*

$$\int_{B_r(p)} |E_g|^{n/2}\, dv_g \leqslant \varepsilon_0, \tag{3.2}$$

then

$$\sup_{B_{r/2}(p)} |\mathrm{Riem}_g| \leqslant \left\{c_1 r^{-4}\Big(\int_{B_r(p)}|E_g|^{n/2}\, dv_g\Big)^{4/n} + c_2\Big(\frac{\mathrm{vol}(S^n(1))}{\mathrm{vol}(M,g)}\Big)^{4/n}\right\}^{1/2}. \tag{3.3}$$

Proof. Since a conformally flat metric of constant scalar curvature is a metric of harmonic curvature (cf. [6, Chapter 16]), we can apply the Bochner-Weitzenböck formula, and using the positivity of R_g then gives

$$\Delta|E_g| \geqslant -c_3|E_g|^2, \tag{3.4}$$

where c_3 is a positive constant depending only on n.

Let $u = |E_g|$. Multiply both sides of (3.4) by $\eta^2 u^\alpha$, where $\alpha \geqslant \frac{1}{2}$ and η is a cut-off function of compact support in $B_r(p)$ which is determined later. Integrating by part, we obtain

$$\begin{aligned}\frac{4\alpha}{(\alpha+1)^2}\int \eta^2 \left|du^{(\alpha+1)/2}\right|^2 dv_g - 2\int \eta u^\alpha |d\eta|\,|du|\, dv_g \\ \leqslant c_3\int \eta^2 u^{\alpha+2} dv_g.\end{aligned} \tag{3.5}$$

The Young inequality implies

$$\eta u^{\alpha}|d\eta||du| \leqslant \varepsilon_n^{-1} u^{\alpha+1}|d\eta|^2 + \varepsilon_n \frac{\eta^2}{(\alpha+1)^2}\left|du^{(\alpha+1)/2}\right|^2, \tag{3.6}$$

where $\varepsilon_n = \frac{3}{4}$ for $n = 3$ and $\varepsilon_n = 1$ for $n \geqslant 4$. Using (3.5) in (3.6) then gives

$$\begin{aligned} &\int \eta^2 \left|du^{(\alpha+1)/2}\right|^2 dv_g \\ &\quad \leqslant \frac{(\alpha+1)^2}{4(\alpha-\varepsilon_n/2)}\Big\{c_3\int \eta^2 u^{\alpha+2} dv_g + 2\varepsilon_n^{-1}\int u^{\alpha+1}|d\eta|^2 dv_g\Big\}. \end{aligned} \tag{3.7}$$

It follows from (2.6) and (3.7) that

$$\begin{aligned} &\Big(\int (\eta u^{(\alpha+1)/2})^{2n/(n-2)} dv_g\Big)^{(n-2)/n} \\ &\quad \leqslant \frac{c_4\alpha}{c_g}\Big\{\int \eta^2 u^{\alpha+2} dv_g + \int |d\eta|^2 u^{\alpha+1} dv_g\Big\} + \frac{1}{\operatorname{vol}(M,g)^{2/n}}\int \eta^2 u^{\alpha+1} dv_g, \end{aligned} \tag{3.8}$$

where c_4 is a positive constant depending only on n.

We first set $\alpha = n/2 - 1$. Then

$$\begin{aligned} &\int \eta^2 u^{(n+2)/2} dv_g \\ &\quad \leqslant \Big(\int_{B_r(p)} u^{n/2} dv_g\Big)^{2/n}\Big(\int (\eta u^{n/4})^{2n/(n-2)} dv_g\Big)^{(n-2)/n}. \end{aligned} \tag{3.9}$$

Taking ε_0 in (3.2) satisfying

$$\varepsilon_0^{2/n} \leqslant \frac{c_g}{(n-2)c_4},$$

from (3.8) and (3.9) we obtain

$$\begin{aligned} &\Big(\int (\eta u^{n/4})^{2n/(n-2)} dv_g\Big)^{(n-2)/n} \\ &\quad \leqslant \frac{c_4(n-2)}{c_g}\int |d\eta|^2 u^{n/2} dv_g + \frac{2}{\operatorname{vol}(M,g)^{2/n}}\int \eta^2 u^{n/2} dv_g \\ &\quad \leqslant c_5 \int (\eta^2 + |d\eta|^2) u^{n/2} dv_g, \end{aligned} \tag{3.10}$$

where

$$c_5 = \max\Big\{\frac{c_4(n-2)}{c_g}, \frac{2}{\operatorname{vol}(M,g)^{2/n}}\Big\}.$$

If we choose the cut-off function η satisfying $\eta = 1$ on $B_{3r/4}(p)$ and $\eta = 0$ on $M\backslash B_r(p)$ with $|d\eta| \leqslant 5/r$, then (3.10) gives

$$\left\|\chi_{B_{3r/4}(p)} u\right\|_{n^2/2(n-2)} \leqslant \Big\{c_5\big(1+\frac{25}{r^2}\big)\varepsilon_0\Big\}^{2/n}, \tag{3.11}$$

where $\chi_{B_{3r/4}(p)}$ and $\|\cdot\|_{n^2/2(n-2)}$ denote the charcteristic function of $B_{3r/4}(p)$ and the $L^{n^2/2(n-2)}$ norm on M, respectively.

Now we reconsider the inequality (3.8) for $\alpha > n/2-1$ and cut-off functions η of compact support in $B_{3r/4}(p)$. From (3.11) and Hölder's inequality, we then obtain

$$\begin{aligned}\int \eta^2 u^{\alpha+2} dv_g &\leqslant \big\|\eta u\big\|_{n^2/2(n-2)} \big\|\eta u^{(\alpha+1)/2}\big\|^2_{2n^2/(n^2-2n+4)} \\ &\leqslant \Big\{c_5(1+\frac{25}{r^2})\varepsilon_0\Big\}^{2/n} \big\|\eta u^{(\alpha+1)/2}\big\|^2_{2n^2/(n^2-2n+4)}.\end{aligned} \tag{3.12}$$

We need the following interpolation inequality in dealing with integral estimates

$$\|f\|_{2n^2/(n^2-2n+4)} \leqslant \varepsilon \|f\|_{2n/(n-2)} + \varepsilon^{-(n-2)/2}\|f\|_2 \tag{3.13}$$

for $f \in L^{2n/(n-2)}(M)$ and $\varepsilon > 0$. It then follows from (3.8), (3.12) and (3.13) that

$$\begin{aligned}&\Big(\int (\eta u^{(\alpha+1)/2})^{2n/(n-2)} dv_g\Big)^{(n-2)/n} \\ &\leqslant c_6\alpha\Big\{\frac{1}{r^{4/n}}(\varepsilon\|\eta u^{(\alpha+1)/2}\|_{2n/(n-2)} + \varepsilon^{-(n-2)/2}\|\eta u^{(\alpha+1)/2}\|_2)^2 \\ &\quad + \int (\eta^2 + |d\eta|^2) u^{\alpha+1} dv_g\Big\},\end{aligned} \tag{3.14}$$

where $c_6 = c_6(n, \mu(M,[g]), \mathrm{vol}(M,g))$. If we choose

$$\varepsilon^2 = \frac{r^{4/n}}{4c_6\alpha},$$

then (3.14) gives

$$\|\eta u^{(\alpha+1)/2}\|^2_{2\chi} \leqslant c_7\Big\{\frac{\alpha^{n/2}}{r^2}\|\eta u^{(\alpha+1)/2}\|_2^2 + \alpha\|(|\eta| + |d\eta|)u^{(\alpha+1)/2}\|_2^2\Big\}, \tag{3.15}$$

where $\chi = n/(n-2)$ and $c_7 = c_7(n, \mu(M,[g]), \mathrm{vol}(M,g))$.

Now we specify the cut-off function η more precisely. Let λ^-, λ^+ be positive constants such that $\frac{1}{2} \leqslant \lambda^- < \lambda^+ \leqslant \frac{3}{4}$ and set $\eta = 1$ on $B_{\lambda^- r}(p), \eta = 0$ on $M \backslash B_{\lambda^+ r}(p)$ with

$$|d\eta| \leqslant \frac{2}{r(\lambda^+ - \lambda^-)}.$$

Let $\Phi(q,r) = (\int_{B_r(p)} u^q dv_g)^{1/q}$. From (3.15) we then obtain

$$\Phi(\chi q, \lambda^- r) \leqslant \Big\{\frac{3c_7 q(1+q^{(n-2)/2})}{r^2(\lambda^+ - \lambda^-)^2}\Big\}^{1/q} \Phi(q, \lambda^+ r) \tag{3.16}$$

for $q \geqslant n/2$. This inequality can be iterated to yield the desired estimate. We set $q_m = \chi^m n/2$ and $\lambda_m^- = \frac{1}{2} + 2^{-(m+3)}, \lambda_m^+ = \frac{1}{2} + 2^{-(m+2)} = \lambda_{m-1}^-$ for $m = 0, 1, 2, \ldots,$

and then by (3.16)

$$\Phi\Big(\chi^m\frac{n}{2},\frac{1}{2}r\Big) \leqslant (c_8 r^{-2})^{\frac{2}{n}\sum_{i=0}^{m-1}\chi^{-i}}(4^{2/n}\chi)^{\sum_{j=0}^{m-1} j\chi^{-j}}\Phi\Big(\frac{n}{2},r\Big) \leqslant c_9 r^{-2}\Phi\Big(\frac{n}{2},r\Big), \tag{3.17}$$

where c_8 and c_9 depend only on $n,\mu(M,[g])$ and $\mathrm{vol}(M,g)$. Letting $m\to\infty$ in (3.17), we have

$$\sup_{B_{r/2}(p)} u \leqslant c_9 r^{-2}\Big(\int_{B_r(p)} u^{n/2}dv_g\Big)^{2/n}. \tag{3.18}$$

Consequently, the inequality (3.3) follows from (3.1) and (3.18). This completes the proof of Proposition 3.2.

4. Proof of Theorem 1.2

In this section we shall sketch the main steps of the proof of Theorem 1.2. We will modify the proof of the convergence theorems for Einstein metrics in [1] and [25].

By Theorem 2.3, if the first case (1°) in Theorem 1.2 does not hold, then there exist subsequence $\{j\}\subset\{i\}$ and a connected compact metric space (M_∞,d_∞) with $\mathrm{diam}(M_\infty,d_\infty)=D_0>0$ such that

$$\lim_{j\to\infty} d_H((M_j,g_j),(M_\infty,d_\infty))=0.$$

Taking a subsequence if necessary, we may assume that

$$\mathrm{diam}(M_j,g_j)\geqslant \tfrac{1}{2}D_0>0 \tag{4.1}$$

for all j and that there exists a $(1/j)$-Hausdorff approximation

$$\varphi_j:(M_j,g_j)\longrightarrow(M_\infty,d_\infty)$$

for each j. For each $p\in M_\infty$, we can find $p_j\in M_j$ such that

$$d_\infty(p,\varphi_j(p_j))<\frac{1}{j}.$$

We define the singular set $\mathcal{S}$ by

$$\mathcal{S}=\bigcap_{0<r<D_0}\Big\{p\in M_\infty;\quad \liminf_{j\to\infty}\int_{B_r(p_j)}|E_{g_j}|^{n/2}dv_{g_j}\geqslant\tilde{\varepsilon}_0 \quad \text{for arbitrary } \{p_j\}_{j=1}^{\infty} \text{ as above}\Big\},$$

where $\tilde{\varepsilon}_0=\tilde{\varepsilon}_0(n,\mu_0)$ denotes a similar constant as ε_0 in Proposition 3.2.

To prove Theorem 1.2, we first note the following.

Lemma 4.1. *$\mathcal{S}$ is a finite set.*

Proof. Take a small constant $r > 0$ and fix it. Then we can cover $\mathcal{S}$ by a finite collection of metric balls $\{B_{2r}(x_a); x_a \in \mathcal{S}\}_{a\in\Lambda}$ with respect to d_∞ such that the collection $\{B_r(x_a)\}_{a\in\Lambda}$ is disjoint. Since $x_a \in \mathcal{S}$, for j sufficiently large there exists a point $x_{a,j} \in M_j$ such that

$$\int_{B_{r/2}(x_{a,j})} |E_{g_j}|^{n/2}\, dv_{g_j} \geqslant \frac{\tilde{\varepsilon}_0}{2}, \tag{4.2}$$

$$d_\infty(x_a, \varphi_j(x_{a,j})) < \frac{1}{j} \quad \text{for } a \in \Lambda, \tag{4.3}$$

$$B_{r/2}(x_{a,j}) \cap B_{r/2}(x_{b,j}) = \phi \quad \text{for } a \neq b. \tag{4.4}$$

It follows from (4.2)–(4.4) that

$$\begin{aligned} \#(\Lambda) &\leqslant 2\tilde{\varepsilon}_0^{-1} \sum_{a\in\Lambda} \int_{B_{r/2}(x_{a,j})} |E_{g_j}|^{n/2}\, dv_{g_j} \\ &\leqslant 2\tilde{\varepsilon}_0^{-1} \int_{M_j} |E_{g_j}|^{n/2}\, dv_{g_j} \leqslant 2R_0\tilde{\varepsilon}_0^{-1}. \end{aligned} \tag{4.5}$$

Since $2R_0\tilde{\varepsilon}_0^{-1}$ is independent of r, letting $r \to 0$ in (4.5), we then obtain

$$\#(\mathcal{S}) \leqslant 2R_0\tilde{\varepsilon}_0^{-1}.$$

This completes the proof of Lemma 4.1.

Now we observe the proof of (2°.2)–(2°.4). Fix a point $p \in M_\infty \backslash \mathcal{S}$.

Lemma 4.2. *There exist r satisfying $0 < r \leqslant \frac{1}{2} d_\infty(p, \mathcal{S})$ and a point $p_j \in M_j$ for each j such that*

$$\sup_{B_{r/2}(p_j)} |\mathrm{Riem}_{g_j}| \leqslant c_{10}(r^{-2} + \mathrm{vol}(S^n(1))^{2/n}), \tag{4.6}$$

where $c_{10} = c_{10}(n, \mu_0)$.

Proof. Taking a subsequence if necessary, we can find r satisfying $0 < r \leqslant \frac{1}{2} d_\infty(p, \mathcal{S})$ and $p_j \in M_j$ for each j such that $d_\infty(p, \varphi_j(p_j)) < 1/j$ and

$$\int_{B_r(p_j)} |E_{g_j}|^{n/2}\, dv_{g_j} \leqslant \tilde{\varepsilon}_0 \quad \text{for all } j. \tag{4.7}$$

It then follows from (3.3) and (4.7) that

$$\begin{aligned} \sup_{B_{r/2}(p_j)} |\mathrm{Riem}_{g_j}| &\leqslant \left\{ \tilde{c}_1 r^{-4} \Big(\int_{B_r(p_j)} |E_{g_j}|^{n/2}\, dv_{g_j} \Big)^{4/n} + c_2 \mathrm{vol}(S^n(1))^{4/n} \right\}^{1/2} \\ &\leqslant \{\tilde{c}_1 r^{-4} \tilde{\varepsilon}_0^{4/n} + c_2 \mathrm{vol}(S^n(1))^{4/n}\}^{1/2} \quad \text{for all } j, \end{aligned}$$

where $\tilde{c}_1 = \tilde{c}_1(n, \mu_0)$ is a similar constant as c_1 in Proposition 3.2. This completes the proof of Lemma 4.2.

From (4.6), we obtain the following estimate of sectional curvature K_{g_j} of (M_j, g_j)

$$|K_{g_j}| \leqslant \kappa_p \qquad \text{on} \quad B_{r_p}(p_j), \tag{4.8}$$

where $r_p = \frac{1}{4} d_\infty(p, \mathcal{S})$ and $\kappa_p = c_{10}((2r_p)^{-2} + \mathrm{vol}(S^n(1))^{2/n})$. The following local injectivity radius estimate is due to Cheeger–Gromov–Taylor [8, Theorem 4.3].

Fact 4.3. *Let $B_r(q)$ be a metric ball of radius r in a Riemannian manifold (M, g) such that for $r' < r$, $\overline{B_{r'}(q)}$ is compact. Assume that on $B_r(q)$, $\omega \leqslant K_g \leqslant \kappa$ and $r \leqslant \pi/\sqrt{\kappa}$ (r arbitrary if $\kappa \leqslant 0$) for some constants ω, κ. Let B_r^ω be a geodesic ball of radius r in the simply-connected space form of constant curvature ω. Then, for positive constants r_0 and s with $r_0 + 2s \leqslant r$ and $r_0 \leqslant r/4$, the injectivity radius $i_q(M, g)$ of (M, g) at q satisfies*

$$i_q(M, g) \geqslant \frac{r_0}{2} \cdot \frac{1}{1 + \mathrm{vol}(B^\omega_{r_0+s})/\mathrm{vol}(B_s(q))}. \tag{4.9}$$

It then follows from (2.4), (4.8) and (4.9) that

$$i_{p_j}(M_j, g_j) \geqslant \frac{l_p}{2} \cdot \frac{1}{1 + \mathrm{vol}(B_{2l_p}^{-\kappa_p})/((5 \cdot 2^{n-2})^{-n/2} \nu_0^{n/2} l_p^n)}, \tag{4.10}$$

where $l_p = \frac{1}{4} \cdot \min\{r_p, \pi/\sqrt{\kappa_p}\}$ and

$$\nu_0 = \frac{n-1}{4(n-2)} \mu_0 > 0.$$

Using (4.8) and (4.10) in the results [17, Chapter 2], for each j there exists a harmonic coordinate system $h_j : B_r(p_j) \longrightarrow \mathbb{R}^n$ with $h_j(B_r(p_j)) \supset B_{2\delta}(0)$ for some $r > 0$ and $\delta > 0$ independent of j such that $\gamma_j = (h_j^{-1})^* g_j$ satisfies the following equation

$$(\gamma_j)^{ik} \partial_i \partial_k \gamma_j + \partial \gamma_j * \partial \gamma_j = -2\mathrm{Ric}_{\gamma_j}, \tag{4.11}$$

where ∂ denotes the differential in this harmonic coordinates and $A * B$ a linear combination of contractions of the tensor product $A \otimes B$ by the metric γ_j. Moreover we have, for $0 < \alpha < 1$,

$$\|\gamma_j\|_{C^{1,\alpha}(B_{2\delta}(0))} \leqslant c_{11}, \tag{4.12}$$

where c_{11} is independent of j. It should be remarked again that a conformally flat Yamabe metric is a metric of harmonic curvature. Then, γ_j also satisfies the following system of quasi-linear elliptic equations

$$\Delta_{\gamma_j} \mathrm{Riem}_{\gamma_j} = \mathrm{Riem}_{\gamma_j} * \mathrm{Riem}_{\gamma_j}. \tag{4.13}$$

It then follows from (4.11)–(4.13), the Schauder interior estimates and the L^p estimates [10] that

$$\|\gamma_j\|_{C^l(B_\delta(0))} \leqslant c_{12}(l) \tag{4.14}$$

for each $l \in \mathbf{N}$, where $c_{12}(l)$ is also independent of j. Taking a subsequence if necessary, we obtain

$$\gamma_j \longrightarrow \gamma_{p,\infty} \quad \text{in the } C^\infty \text{ topology on } B_\delta(0),$$

where $\gamma_{p,\infty}$ is a conformally flat metric of positive constant scalar curvature on $B_\delta(0)$.

Now $\varphi_j \circ h_j^{-1}$ is a $(1/j)$-Hausdorff approximation from $(B_\delta(0), \gamma_j)$ to a neighborhood of p equipped with d_∞. Then it converges to an isometry $H_p : (B_\delta(0), \gamma_{p,\infty}) \longrightarrow (U_p, d_\infty)$, where U_p is also a neighborhood of p. Moreover, for any $p, q \in M_\infty \backslash \mathcal{S}$, $H_q^{-1} \circ H_p : (H_p^{-1}(U_p \cap U_q), \gamma_{p,\infty}) \longrightarrow (H_q^{-1}(U_p \cap U_q), \gamma_{q,\infty})$ is also an isometry unless $U_p \cap U_q = \phi$. In particular it is differentiable. Thus $\{H_p\}_{p \in M_\infty \backslash \mathcal{S}}$ and $\{\gamma_{p,\infty}\}_{p \in M_\infty \backslash \mathcal{S}}$ give a coordinate system on $M_\infty \backslash \mathcal{S}$ and a conformally flat metric g_∞ of positive constant scalar curvature on $M_\infty \backslash \mathcal{S}$ respectively.

For the proof of (2°.3), we recall the following local version of Gromov's compactness theorem [1, Theorem 2.2].

Fact 4.4. *Let $\{(V_i, h_i)\}_{i \in \mathbf{N}}$ be a sequence of closed Riemannian n-manifolds and D_i a domain of V_i with smooth boundary ∂D_i for each i. Suppose that for all i*

(i) $|\nabla^l \mathrm{Riem}_{h_i}|(x) \leqslant \Lambda(l)$ *for* $l \geqslant 0$,

(ii) $i_x(V_i, h_i) \geqslant \Lambda_1 > 0$,

(iii) $0 < \Lambda_2 \leqslant \mathrm{vol}(D_i, h_i) \leqslant \Lambda_3$

for $x \in D_i$. For a given positive constant ε, let $D_i(\varepsilon) = \{x \in D_i ; \mathrm{dist}_{(V_i,h_i)}(x, \partial D_i) > \varepsilon\}$ (assumed nonempty). Then there exist a subsequence $\{j\} \subset \{i\}$, a smooth Riemannian n-manifold $(D_\infty(\varepsilon), h_\infty)$ and a diffeomorphism $F_j : D_\infty(\varepsilon) \longrightarrow D_j(\varepsilon)$ for each j such that $(F_j)^* h_j$ converges to h_∞ in the C^∞ topology on $D_\infty(\varepsilon)$.

For each $x_a \in \mathcal{S} = \{x_1, \cdots, x_k\}$, let $\{x_{a,j}\}_{j \in \mathbf{N}}$ be as in the proof of Lemma 4.1. For each $m \in \mathbf{N}$, define the open subsets $\Omega_j(2^{-m})$ in M_j and $\Omega_\infty(2^{-m})$ in M_∞ by

$$\Omega_j(2^{-m}) = \{p \in M_j ;\ \mathrm{dist}_{(M_j,g_j)}(p, x_{a,j}) > 2^{-m} \text{ for } a = 1, \cdots, k\}$$

and

$$\Omega_\infty(2^{-m}) = \{p \in M_\infty ; d_\infty(p, \mathcal{S}) > 2^{-m}\}$$

respectively. Modfying $\Omega_j(2^{-m})$ if necessary, we may assume that each $\Omega_j(2^{-m})$ has a smooth boundary. From (4.10) and (4.14), we obtain for all j

$$|\nabla^l \mathrm{Riem}_{g_j}|(x) \leqslant c_{13}(l, m) \qquad \text{for} \quad l \geqslant 0, \tag{4.15}$$

$$i_x(M_j, g_j) \geqslant c_{14}(m) > 0 \tag{4.16}$$

for $x \in \Omega_j(2^{-m})$, where $c_{13}(l,m)$ and $c_{14}(m)$ are independent of j. Also from (2.4), (4.1) and that $\mathrm{vol}(M_j, g_j) = 1$, we have for all j

$$0 < c_{15}(m) \leqslant \mathrm{vol}(\Omega_j(2^{-m}), g_j) \leqslant 1 \tag{4.17}$$

for some constant $c_{15}(m)$ independent of j. Replace D_i and $D_i(\varepsilon)$ in Fact 4.4 by each component of $\Omega_j(2^{-m-1})$ and $\Omega_j(2^{-m})$ respectively. It then follows from (4.15)-(4.17) that, for each m, there exist a subsequence $\{j_m\} \subset \{j\}$, a smooth Riemannian manifold $(\Omega_\infty^m, g_\infty^m)$ and a diffeomorphism $F_{j_m} : \Omega_\infty^m \longrightarrow \Omega_{j_m}(2^{-m})$ for each j_m such that $(F_{j_m})^* g_{j_m}$ converges to g_∞^m in the C^∞ topology on Ω_∞^m. Moreover we assume $\{j_{m+1}\} \subset \{j_m\}$ for all m.

Now we remark that $\varphi_{j_m} \circ F_{j_m} : \Omega_\infty^m \longrightarrow M_\infty$ converges to an isometry $G^m : (\Omega_\infty^m, g_\infty^m) \longrightarrow (\Omega_\infty(2^{-m}), g_\infty)$ for each m. Take the diagonal sequence $\{j_j\}$ of $\{j_m\}_{j,m\in\mathrm{N}}$. We shall rewrite the index "j_j" by "j" again. Then we obtain that, for each m, there exists an into diffeomorphism $\Phi_j^m : \Omega_\infty(2^{-m}) \longrightarrow M_j$ for j sufficiently large such that $(\Phi_j^m)^* g_j$ converges to g_∞ in the C^∞ topology on $\Omega_\infty(2^{-m})$. For a compact subset $K \subset M_\infty \backslash \mathcal{S}$, there exists $m \in \mathrm{N}$ such that $K \subset \Omega_\infty(2^{-m})$. Thus we can take $\Phi_j = \Phi_j^m|_K$ as in (2°.3).

Finally we observe the proof of (2°.5) and (2°.6) briefly. Fix a point $x_a \in \mathcal{S}$. There exists $x_{a,j} \in M_j$ such that $d_\infty(\varphi_j(x_{a,j}), x_a) < 1/j$. Since $\mathcal{S}$ is a finite set, we can take $\delta > 0$ so that $(B_{2\delta}(x_a)\backslash\{x_a\}) \cap \mathcal{S} = \phi$. For each j, we define the positive number r_j in (2°.5) by

$$r_j = \sup_{B_\delta(x_{a,j})} |E_{g_j}|.$$

By the definition of $\mathcal{S}$,

$$r_j \longrightarrow \infty \qquad \text{as } j \longrightarrow \infty. \tag{4.18}$$

Moreover we may assume that $|E_{g_j}|$ takes a local maximum value r_j at $x_{a,j}$. We consider the new sequence of pointed Riemannian manifolds $((M_j, \tilde{g}_j), x_{a,j})$, where $\tilde{g}_j = r_j g_j$. This sequence satisfies:

$$\text{Each } \tilde{g}_j \text{ is a conformally flat metric of constant scalar curvature,} \tag{4.19}$$

$$\sup_{B_{\sqrt{r_j}\cdot\delta}(x_{a,j}, \tilde{g}_j)} |E_{\tilde{g}_j}| = 1, \qquad |E_{\tilde{g}_j}|(x_{a,j}) = 1, \tag{4.20}$$

$$R_{\tilde{g}_j} = r_j^{-1} R_{g_j} = r_j^{-1} \mu(M_j, [g_j]), \tag{4.21}$$

$$\int_{B_{\sqrt{r_j}\cdot\delta}(x_{a,j}, \tilde{g}_j)} |E_{\tilde{g}_j}|^{n/2} dv_{\tilde{g}_j} = \int_{B_\delta(x_{a,j}, g_j)} |E_{g_j}|^{n/2} dv_{g_j}, \tag{4.22}$$

$$\mathrm{diam}(M_j, \tilde{g}_j) \longrightarrow \infty \qquad \text{as } j \longrightarrow \infty, \tag{4.23}$$

$$\mathrm{vol}(M_j, \tilde{g}_j) \longrightarrow \infty \qquad \text{as } j \longrightarrow \infty. \tag{4.24}$$

Using Fact 4.5 below due to Gromov [12], and then choosing a subsequence if necessary, we have

$$((M_j, \tilde{g}_j), x_{a,j}) \longrightarrow \text{a pointed metric space } ((N_a, d_a), x_{a,\infty})$$

in the pointed Hausdorff distance.

Fact 4.5. *Let $\{((M_i, g_i), p_i)\}_{i=1}^{\infty}$ be a sequence of pointed complete Riemannian n-manifolds. Suppose that for every $r > 0$ there exists a positive constant $C = C(r)$ such that*

$$\mathrm{Ric}_{g_i} \geqslant -C \qquad \text{on } \overline{B_r(p_i)}$$

for all i. Then the closure of $\{((M_i, g_i), p_i)\}_{i=1}^{\infty}$ is compact in $\mathcal{MET}_0$ with respect to the pointed Hausdorff distance.

By the same argument as above, we can show that N_a has a structure of smooth n-manifold and a smooth Riemannian metric h_a on N_a compatible with d_a, which satisfies the property in (2°.5d). Moreover it follows from (2.4) and (4.18)-(4.24) that $((N_a, h_a), x_{a,\infty})$ satisfies the property in (2°.5c).

The proof of (2°.6) easily follows from the lower semicontinuity of the curvature integral.

5. Applications

In this section we shall give two applications of Theorem 1.2.

We first prove the following.

Corollary 5.1. *Let $\{(M_i, g_i)\}_{i=1}^{\infty}$ be a sequence of compact conformally flat 4-manifolds with unit volume which satisfy*

$$\mu(M_i, [g_i]) \geqslant \mu_0 > 0, \tag{5.1}$$

$$-\chi(M_i) \leqslant m_0 \ (\in \mathbb{Z}) \tag{5.2}$$

for some constants μ_0, m_0, where $\chi(M_i)$ denotes the Euler number of M_i. Then there exists a Yamabe metric $\bar{g}_i \in [g_i]$ with unit volume for each i such that the same conclusion as in Theorem 1.2 holds for the sequence $\{(M_i, \bar{g}_i)\}_{i=1}^{\infty}$.

Proof. By applying the 4-dimensional Gauss-Bonnet formula to a compact conformally flat 4-manifold (M, g),

$$16\pi\chi(M) = \frac{1}{12}\int_M R_g^2\, dv_g - \int_M \left|E_g\right|^2 dv_g. \tag{5.3}$$

Since the Yamabe problem had been solved affirmatively, we can find a Yamabe metric $\bar{g}_i \in [g_i]$ of unit volume for each i. Using (2.1), (2.3), (5.1) and (5.2) in (5.3) then gives

$$\begin{aligned}\int_{M_i} \left|E_{\bar{g}_i}\right|^2 dv_{\bar{g}_i} &= -16\pi^2\chi(M_i) + \tfrac{1}{12}\mu(M_i, [\bar{g}_i])^2 \\ &\leqslant 16\pi^2 m_0 + 12\mathrm{vol}(S^4(1))\end{aligned} \tag{5.4}$$

for all i. The assertion is immediate from (5.1), (5.4) and Theorem 1.2. This completes the proof of Corollary 5.1.

Finally we prove the following gap theorem (see Izeki [15] for $n = 4$).

Theorem 5.2. *For a positive constant μ_0, there exists a positive constant $\delta_0 = \delta_0(n, \mu_0)$ such that each element $(M, g) \in \mathcal{CY}(n, \mu_0, \delta_0)$ is a spherical space form. Moreover,*

$$\#(\pi_1(M)) \leqslant \left(\frac{n(n-1)\mathrm{vol}(S^n(1))^{2/n}}{\mu_0}\right)^{n/2} \tag{5.5}$$

for all $(M, g) \in \mathcal{CY}(n, \mu_0, \delta_0)$.

Proof. Using a similar argument as in the proof of Proposition 3.2, then we can find positive constants $\tilde{\delta}_0 = \tilde{\delta}_0(n, \mu_0)$ and $c_{10} = c_{10}(n, \mu_0)$ such that each element $(M, g) \in \mathcal{CY}(n, \mu_0, \tilde{\delta}_0)$ satisfies

$$\sup_M |E_g| \leqslant c_{10}\Big(\int_M |E_g|^{n/2} dv_g\Big)^{2/n} \tag{5.6}$$

$$\begin{aligned} \sup_M |\mathrm{Riem}_g| &\leqslant c_{10}\Big\{\Big(\int_M |E_g|^{n/2} dv_g\Big)^{n/2} + \mathrm{vol}(S^n(1))^{2/n}\Big\} \\ &\leqslant c_{10}(\tilde{\delta}_0^{2/n} + \mathrm{vol}(S^n(1))^{2/n}). \end{aligned} \tag{5.7}$$

Our assertion will be done by contradiction. We consider the sequences $\{(M_i, g_i)\}_{i=1}^{\infty}$, each of which is not a spherical space form, and $(\tilde{\delta}_0 \geqslant)\delta_1 > \delta_2 > \cdots \to 0$ such that for each i

$$(M_i, g_i) \in \mathcal{CY}(n, \mu_0, \delta_i). \tag{5.8}$$

From (5.7) and the well-known Bishop comparison theorem [7, Corollary 4, p.245], there exists a positive constant $D_0 = D_0(n, \mu_0, \tilde{\delta}_0)$ such that

$$\mathrm{diam}(M_i, g_i) \geqslant D_0 > 0 \quad \text{for all } i. \tag{5.9}$$

We now apply Theorem 1.2 to the sequence $\{(M_i, g_i)\}_{i=1}^{\infty}$, and then from (5.7)–(5.9) there exist a subsequence $\{j\} \subset \{i\}$ and a conformally flat smooth n-manifold (M_∞, g_∞) such that

$$\lim_{j \to \infty} d_H((M_j, g_j), (M_\infty, g_\infty)) = 0.$$

Moreover Greene-Wu's proof of Gromov's convergence theorem [11] (see also Kasue [18] and Peters [27]) implies that

> There exists a diffeomorphism $\Phi_j : M_\infty \longrightarrow M_j$ for each j such that $(\Phi_j)^* g_j$ converges to g_∞ in the C^∞ topology on M_∞. (5.10)

Note that, from (5.6) and (5.10), (M_∞, g_∞) is a compact conformally flat Einstein manifold of positive scalar curvature, and hence (M_∞, g_∞) is a spherical space form.

Let $(\tilde{M}_j, \tilde{g}_j)$ denote the universal covering of (M_j, g_j) for each j. Since each $(\tilde{M}_j, \tilde{g}_j)$ is conformally flat and $\tilde{M}_j$ is diffeomorphic to $S^n(1)$, we can apply Kuiper's uniformization theorem [23], and then $(\tilde{M}_j, \tilde{g}_j)$ is conformally diffeomorphic to $S^n(1)$ for all j. By applying Obata's uniqueness theorem [26] (see also [31]) to each constant scalar curvature metric $\tilde{g}_j$, we obtain that $(\tilde{M}_j, \tilde{g}_j)$ is homothetic to $S^n(1)$ for all j. It contradicts the fact that (M_j, g_j) is not a spherical space form. This completes the proof of the first assertion.

Fix an element $(M, g) \in \mathcal{CY}(n, \mu_0, \delta_0)$ and let $\gamma = \#(\pi_1(M))$. It then follows from (2.1) that

$$\mu_0 \leqslant \mu(M, [g]) = n(n-1)\Big(\frac{\mathrm{vol}(S^n(1))}{\gamma}\Big)^{2/n}.$$

This completes the proof of Theorem 5.2.

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